*Int. J. Solids Structures* Vol. 25, No. 10, pp. 1201–1219, 1989
Printed in Great Britain.

0020–7683/89 $3.00 + .00


# DILATATIONALLY NONLINEAR ELASTIC MATERIALS—I. SOME THEORY†

ROHAN ABEYARATNE and GUO-HUA JIANG
Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

(*Received* 28 *October* 1988; *in revised form* 3 *March* 1989)

**Abstract**—This paper, which is the first in a two-part study, addresses certain issues concerning the small-strain theory of nonlinear elasticity. It considers isotropic materials which possess a linear response in shear and a nonlinear response in dilatation, and (i) establishes an explicit necessary and sufficient condition for the existence of piecewise homogeneous deformations, (ii) obtains a characterization of the set of all such deformations, (iii) derives an expression for the "driving traction" on a surface of strain discontinuity, (iv) discusses the notion of a kinetic law, and finally (v) makes some remarks on (a) the driving force on a crack-tip, (b) the uncoupling of the shear and dilatational invariants in the strain energy function, and (c) the intersection of a surface of strain discontinuity with a traction-free surface. While the analysis is carried out within a three-dimensional setting, the results are shown to have a particularly simple form when expressed in terms of a certain constitutive function $\Sigma(\varepsilon)$. In Part II of this study we examine a specific boundary-value problem.

## 1. INTRODUCTION

In this paper, which is the first in a two-part study, we show that certain features of the finite theory of elasticity are also present in the small-strain nonlinear theory; the particular class of constitutive laws that we consider here is one that has been used to model the mechanical response of ceramic composites undergoing supercritical phase transformations. In Part II we will examine a specific boundary-value problem.

A number of recent studies in *finite deformation* elasticity theory have been concerned with "nonelliptic materials", see for example Abeyaratne (1980, 1983), Ericksen (1975), Gurtin (1983), Hutchinson and Neale (1982), James (1984, 1986), Knowles (1979), Knowles and Sternberg (1978) and Silling (1987). Such materials are capable of sustaining deformations whose gradient is discontinuous across certain surfaces in the body; this leads to a tremendous lack of uniqueness of solution to boundary-value problems, since the class of functions from among which a solution is sought has to be greatly enlarged to allow for such deformations. Moreover, quasi-static motions of a body composed of such a nominally elastic material can involve a dissipation of mechanical energy at particles located on a moving surface of discontinuity (Knowles, 1979).

Continuum mechanical treatments of stress-induced phase transformations in solids involve such deformations (e.g. James, 1984, 1986). In the context of phase transformations, a surface of displacement gradient discontinuity corresponds to a phase boundary separating two different phases of the material, and the aforementioned non-uniqueness might be thought of as arising due to the fact that the classical equations of the continuum theory do not account for the kinetics of the transformation.

In the present study we examine the corresponding issues within the *infinitesimal* strain theory of nonlinear elasticity. We show that the aforementioned phenomena (of discontinuous, dissipative, non-unique deformations) persist in the infinitesimal strain theory too, suggesting that (in some sense) it is the constitutive nonlinearity rather than the kinematical one that is the principal source of these features.

In this study we restrict attention to the particular class of constitutive laws that were proposed by Budiansky, Hutchinson and Lambropoulos (1983) for modeling the mechanical response of certain transforming ceramics. The fracture toughness of these

† The results reported here were obtained in the course of an investigation supported in part by the U.S. Office of Naval Research.



SAS 25:10-G

ceramic composites (which contain second phase particles that undergo a phase transformation) was known to be higher than that of the brittle ceramic matrix (Garvie *et al.*, 1975; Evans and Cannon, 1986; McMeeking and Evans, 1982). In order to model this phenomenon at the continuum level, Budiansky *et al.* (1983) derived a homogenized constitutive law for such composites using arguments based on the self-consistent method. They argued that since the transformation leads to particles twinned into layers of alternating shear, the average shear associated with the transformation, from a continuum point of view, is essentially zero. Accordingly, they proposed (and studied) a constitutive law with a linear response in shear and a tri-linear response in dilatation; see also Silling (1987). It is this class of materials that we will study here (modified to allow the dilatational response to be arbitrary).

Chen and Reyes Moral (1986) have experimentally examined the relative importance of shear and dilatation in transforming ceramics, and Lambropoulos (1986) has proposed a more general constitutive law that accounts for both of these effects. We do not consider such generalizations here.

In this paper, we first recall the ellipticity conditions for the three-dimensional displacement equations of equilibrium; they are shown to have a particularly simple interpretation in terms of the stress response function of the material in uni-axial deformation, $\Sigma(\varepsilon)$. Next, in Section 3, we examine conditions under which a three-dimensional piecewise homogeneous deformation can be sustained by the material, and derive a single necessary and sufficient condition for the existence of piecewise homogeneous deformations. This condition too is expressed in a particularly simple form in terms of $\Sigma(\varepsilon)$; in addition to providing information on existence, it also allows us to characterize the set of *all* possible piecewise homogeneous equilibrium states.

As show by Knowles (1979), when the theory of finite elasticity is broadened to allow for equilibrium fields with discontinuous displacement gradients, the usual balance between the rate of external work and the rate of storage of elastic energy during a quasi-static motion no longer holds. Instead, one finds that mechanical energy may be dissipated at points on the surfaces of discontinuity. This in turns permits one to introduce the notion of a "*driving traction*" which may be viewed as a normal traction that the body applies to the surface of discontinuity at each of its points. In Section 4 we observe that a disssipation of mechanical energy can also occur in the small-strain theory of elasticity, and we derive an explicit expression for the driving traction in the case of the aforementioned materials; this too may be simply expressed in terms of $\Sigma(\varepsilon)$.

The stress response function in uni-axial deformation $\Sigma(\varepsilon)$ plays such a visible role in all of these results because (as shown in Section 3) the local deformations on the two sides of a surface of discontinuity differ from each other by precisely a uni-axial stretch in the direction normal to that surface.

In Section 5 we briefly discuss the need for additional constitutive information in order to complete the theory. As discussd there, this might, for example, take the form of a "kinetic law" which relates the driving traction on the surface of discontinuity to its velocity of propagation. The "flow rule" utilized by Budiansky *et al.* (1983) is equivalent to a particular kinetic law as will be discussed more fully in Part II.

The results in this paper pertaining to the existence of piecewise homogeneous deformations (in three dimensions) have a similar form to analogous results for isotropic, incompressible elastic materials undergoing finite *plane* deformations (Abeyaratne and Knowles, 1989). Likewise, the (three-dimensional) driving traction formula here is similar to the corresponding formulae for finite *plane* and *anti-plane* deformations (Abeyaratne and Knowles, 1989; Yatomi and Nishimura, 1983). A discussion of kinetic relations in the particular setting of the one-dimensional theory of bars was given in Abeyaratne and Knowles (1989).

Finally we make some closing remarks in Section 6. In Section 6.1 we observe that the driving force on the tip of a crack is generally affected by the presence of a surface of strain discontinuity, *even if the crack is stationary*; see Budiansky *et al.* (1983). A relationship between the far field value of the $J$-integral, the near-tip value of $J$ and the resultant driving force on the surface of discontinuity is derived. The possibility of sustaining anti-plane shear deformations is discussed in Section 6.2 and is connected to the coupling of the shear and dilatational terms in the strain energy function. In Section 6.3 we show that when a surface of strain discontinuity intersects a traction-free surface, it must do so tangentially (unless the strength of the discontinuity vanishes at that point).

## 2. PRELIMINARIES

Consider an elastic body occuping a region $R$ of three-dimensional space. Let $\mathbf{x}$ be the position vector of a particle in $R$ and let $\mathbf{u}(\mathbf{x})$ be its displacement. Suppose that there is a smooth surface $S$ which lies in $R$, such that the displacement field is continuous on $R$ and twice continuously differentiable on $R-S$; $\nabla\mathbf{u}$ may suffer a finite jump discontinuity across $S$. Let $\mathbf{H}$, $\boldsymbol{\varepsilon}$ and $\mathbf{e}$ denote the displacement gradient tensor, the infinitesimal strain tensor and the strain deviator respectively:

$$\left.\begin{aligned} H_{ij} &= u_{i,j} \\ \varepsilon_{ij} &= 1/2(u_{i,j}+u_{j,i}) \\ e_{ij} &= \varepsilon_{ij}-1/3\varepsilon_{kk}\delta_{ij} \end{aligned}\right\} \quad \text{for } x \in R-S. \tag{1}$$

Displacement continuity across $S$ requires

$$[[u_{i,j}]]\ell_j = 0 \quad \text{for} \quad \mathbf{x}\in S \tag{2}$$

for all vectors $\ell$ that are tangential to $S$ at $\mathbf{x}$; $[[\cdot]]$ indicates the jump across the surface $S$. Finally, let $\Delta(\mathbf{x})$ and $k(\mathbf{x})$ denote the respective strain invariants which represent the dilatation and shear at a particle $\mathbf{x}$:

$$\left.\begin{aligned} \Delta &= \operatorname{tr}\boldsymbol{\varepsilon} \\ k &= [2\operatorname{tr}(\mathbf{e}^2)]^{1/2} \end{aligned}\right\} \quad \text{for } \mathbf{x}\in R-S. \tag{3}$$

Next, let $\boldsymbol{\sigma}(\mathbf{x})$ be the stress tensor field on $R$ and suppose that $\boldsymbol{\sigma}(\mathbf{x})$ is continuously differentiable on $R-S$; $\boldsymbol{\sigma}$ may suffer a finite jump discontinuity across $S$. Equilibrium in the absence of body forces requires

$$\sigma_{ij,j} = 0, \quad \sigma_{ij} = \sigma_{ji} \quad \text{for } \mathbf{x}\in R-S \tag{4}$$

$$[[\sigma_{ij}]]n_j = 0 \quad \text{for } \mathbf{x}\in S \tag{5}$$

where $\mathbf{n}$ is a unit normal vector on $S$. A surface $S$ which carries jump discontinuities in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ while maintaining displacement and traction continuity is called an *equilibrium shock* or *phase boundary*.

Turning to the constitutive law of the material at hand, suppose that it is homogeneous, isotropic and hyperelastic. The elastic potential $W$ then depends on the deformation only through the three principal invariants of strain. A particular case of special interest is that in which $W$ depends only on the shear and dilatational invariants $k$ and $\Delta$:

$$W(\boldsymbol{\varepsilon}) = W(k,\Delta). \tag{6}$$

The stress–strain relation $\boldsymbol{\sigma} = \partial W/\partial\boldsymbol{\varepsilon}$ at a particle $\mathbf{x}\in R-S$ then specializes to

$$\sigma_{ij} = (2/k)\,\partial W/\partial k\,\varepsilon_{ij} + (\partial W/\partial\Delta - (2\Delta/3k)\partial W/\partial k)\,\delta_{ij}. \tag{7}$$

If the material is such that the mean stress $\sigma_{ii}/3$ depends on the deformation only through the dilatation $\varepsilon_{ii}$, one can show using (7) that it is necessary and sufficient that (6) have the separable form $W(k,\Delta) = f(k)+g(\Delta)$ which can be more conveniently written as

$$W(k,\Delta) = \int_0^k \hat{\tau}(\kappa)\,d\kappa + \int_0^\Delta \hat{\sigma}(\xi)\,d\xi \quad \text{for } k \geqslant 0,\ -\infty < \Delta < \infty;\quad \hat{\tau}(0) = \hat{\sigma}(0) = 0. \tag{8}$$

(Alternatively, one show that the components of deviatoric stress depend on the deformation solely through the components of deviatoric strain if and only if $W$ has the form (8).) The constitutive functions $\hat{\tau}(k)$ and $\hat{\sigma}(\Delta)$ may be readily interpreted as follows: in a simple shear deformation $u_1 = kx_2$, $u_2 = 0$, $u_3 = 0$, the shear stress component $\sigma_{12}$ is found from (8), (7) to be $\sigma_{12} = \hat{\tau}(k)$; in a pure dilatational deformation $u_i = (\Delta/3)x_i$, one finds that $\sigma_{ii}/3 = \hat{\sigma}(\Delta)$. Thus, the function $\hat{\tau}(k)$ is the *shear stress response function of the material in simple shear*, while the function $\hat{\sigma}(\Delta)$ is the *mean stress response function of the material in pure dilatation*.

Finally, we further specialize (8) to the case in which the shear stress response in simple shear is linear: $\hat{\tau}(k) = \mu k$. This is motivated by the fact that such constitutive relations appear to be of interest in the continuum mechanical modeling of certain ceramic composites containing particles which undergo stress induced phase transformations (see Budiansky *et al.*, 1983; Evans and Cannon, 1986). Thus, in this study we consider materials characterized by an elastic potential

$$W(k,\Delta) = (\mu/2)k^2 + \int_0^\Delta \hat{\sigma}(\xi)\,d\xi \quad \text{for } k \geqslant 0,\ -\infty < \Delta < \infty \tag{9}$$

where $\mu$ ($>0$) is the infinitesimal shear modulus of the material. The stress–strain relation (7) now specializes to

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\hat{\sigma}(\Delta) - 2\mu\Delta/3)\delta_{ij}. \tag{10}$$

The bulk modulus of the material (9) is

$$B(\Delta) \equiv \hat{\sigma}(\Delta)/\Delta \quad \text{for } -\infty < \Delta < \infty. \tag{11}$$

It is useful for later purposes to consider the response of this body in a uni-axial deformation $u_1 = \varepsilon x_1$, $u_2 = u_3 = 0$. From (10) one gets $\sigma_{11} = \Sigma(\varepsilon)$ where

$$\Sigma(\varepsilon) \equiv \hat{\sigma}(\varepsilon) + 4\mu\varepsilon/3 \quad \text{for } -\infty < \varepsilon < \infty; \tag{12}$$

$\Sigma(\varepsilon)$ is the *stress response function of the material in uni-axial deformation*.

The displacement equations of equilibrium for the class of materials under discussion here are, by (10), (4), (1), (3)

$$c_{ijkl}(\boldsymbol{\varepsilon})u_{k,jl} = 0 \quad \text{for } \mathbf{x} \in R - S \tag{13}$$

where

$$c_{ijkl}(\boldsymbol{\varepsilon}) = \mu(\delta_{ik}\delta_{lj} + \delta_{kj}\delta_{li}) + (\hat{\sigma}'(\Delta) - 2\mu/3)\delta_{ij}\delta_{kl}. \tag{14}$$

The system of partial differential equations (13) is said to be (strongly) *elliptic* at a solution $\mathbf{u}$ and at a point $\mathbf{x} \in R - S$ if

$$c_{ijkl}(\boldsymbol{\varepsilon}(\mathbf{x}))m_i n_j m_k n_l > 0 \tag{15}$$

for all unit vectors $\mathbf{m}$ and $\mathbf{n}$. It is not difficult to show from (14), (15) and $\mu > 0$ that (strong) ellipticity prevails if and only if

$$\hat{\sigma}'(\Delta(\mathbf{x})) > -4\mu/3 \tag{16}$$

where $\Delta(\mathbf{x}) = \varepsilon_{kk}(\mathbf{x})$ is the dilatation associated with the given deformation at the point under consideration. Observe from (12) that this ellipticity condition can be expressed simply in terms of the stress response function in uni-axial deformations as

$$\Sigma'(\Delta(\mathbf{x})) > 0. \tag{17}$$

Thus the ellipticity of the governing equations is directly related to the invertibility of the stress response function in uni-axial deformations. If $\Sigma$ fails to be monotonically increasing on $-\infty < \varepsilon < \infty$, ellipticity will be lost at some deformation. If $\Sigma'(\varepsilon) > 0$ for all $\varepsilon$, we say that the *material* is elliptic. We assume throughout that $\Sigma'(0) > 0$ so that ellipticity prevails at the undeformed state; since $\Sigma'(0) = \kappa + 4\mu/3$ where $\kappa$ is the infinitesimal bulk modulus, this, together with $\mu > 0$, are the usual ellipticity conditions of linear elasticity.

## 3. PIECEWISE HOMOGENEOUS DISPLACEMENT FIELDS

Not all homogeneous, isotropic elastic materials characterized by the constitutive relation (10) can sustain deformations with discontinuous strains. In this section, we determine a simple necessary and sufficient condition on the material which determines whether or not it can sustain piecewise homogeneous deformations of this type. In addition, for materials that can sustain such deformations, we obtain a characterization (in a certain sense) of the entire collection of possible piecewise homogeneous deformations.

We now consider the special case in which $R$ coincides with all of $(x_1, x_2, x_3)$-space, $S$ is a plane through the origin, and the displacement gradient is constant on each side of $S$. Let $\mathbf{n}$ be a unit vector normal to the plane $S$, and let $\overset{+}{R}$, $\overset{-}{R}$ be the two open half-spaces into which $S$ divides $R$ with the normal $\mathbf{n}$ pointing into $\overset{+}{R}$. The field equations (13) will then be trivially satisfied in $R-S$, and all that remains to be fulfilled are the jump conditions (2) and (5).

Consider the piecewise homogeneous displacement field

$$\mathbf{u} = \begin{cases} \overset{+}{\mathbf{H}}\mathbf{x} & \text{for } \mathbf{x} \in \overset{+}{R} \\ \overset{-}{\mathbf{H}}\mathbf{x} & \text{for } \mathbf{x} \in \overset{-}{R} \end{cases} \tag{18}$$

where the displacement gradient tensors $\overset{+}{\mathbf{H}}$ and $\overset{-}{\mathbf{H}}$ are constant and distinct:

$$\overset{+}{\mathbf{H}} \neq \overset{-}{\mathbf{H}}. \tag{19}$$

Define $\overset{+}{\boldsymbol{\varepsilon}}$, $\overset{-}{\boldsymbol{\varepsilon}}$, $\overset{+}{\Delta}$ and $\overset{-}{\Delta}$ by

$$\overset{+}{\varepsilon}_{ij} = 1/2(\overset{+}{H}_{ij} + \overset{+}{H}_{ji}), \quad \overset{-}{\varepsilon}_{ij} = 1/2(\overset{-}{H}_{ij} + \overset{-}{H}_{ji}) \tag{20}$$

$$\overset{+}{\Delta} = \overset{+}{\varepsilon}_{kk}, \quad \overset{-}{\Delta} = \overset{-}{\varepsilon}_{kk}. \tag{21}$$

The displacement field (18) will be continuous across $S$ if and only if

$$\overset{+}{H}_{ij}\ell_j = \overset{-}{H}_{ij}\ell_j \quad \text{for all unit vectors } \boldsymbol{\ell} \text{ normal to } \mathbf{n} \tag{22}$$

while by (5), (10) the tractions will be continuous across $S$ if and only if

$$2\mu\overset{+}{\varepsilon}_{ij}n_j + (\hat{\sigma}(\overset{+}{\Delta}) - 2\mu\overset{+}{\Delta}/3)n_i = 2\mu\overset{-}{\varepsilon}_{ij}n_j + (\hat{\sigma}(\overset{-}{\Delta}) - 2\mu\overset{-}{\Delta}/3)n_i. \tag{23}$$

Given a tensor $\overset{+}{\mathbf{H}}$, the *shock problem* consists of finding a tensor $\overset{-}{\mathbf{H}}$ and a unit vector $\mathbf{n}$ such that (22) and (23) (with (20), (21)), hold.

We first establish a necessary condition which must hold if the shock problem is to have a solution. To this end, suppose that given $\overset{+}{\mathbf{H}}$, there is a tensor $\overset{-}{\mathbf{H}}$ and a unit vector $\mathbf{n}$

such that (20)–(23) hold. It can be readily shown that (22) holds if and only if there exists a vector **a** such that

$$\bar{H}_{ij} = \overset{+}{H}_{ij} + a_i n_j. \tag{24}$$

Thus (20), (21), (24) yield

$$\bar{\varepsilon}_{ij} = \overset{+}{\varepsilon}_{ij} + 1/2(a_i n_j + a_j n_i), \tag{25}$$

$$\bar{\Delta} = \overset{+}{\Delta} + a_i n_i. \tag{26}$$

Turning next to the requirement (23) and multiplying it by the components $\ell_i$ of any unit vector normal to **n** gives

$$\overset{+}{\varepsilon}_{ij} \ell_i n_j = \bar{\varepsilon}_{ij} \ell_i n_j \tag{27}$$

which in view of (25) simplifies to

$$a_i \ell_i = 0. \tag{28}$$

Since this must hold for all unit vectors $\ell$ in the plane $S$, it follows that **a** is parallel to **n**:

$$\mathbf{a} = \alpha \mathbf{n}. \tag{29}$$

By (29), (26),

$$\alpha = \Delta - \overset{+}{\Delta}. \tag{30}$$

Moreover (25) can be written, in view of (29), as

$$\varepsilon_{ij} = \overset{+}{\varepsilon}_{ij} + \alpha n_i n_j. \tag{31}$$

Finally, multiply the traction continuity condition (23) by $n_i$ and use (31), (30) to obtain

$$\hat{\sigma}(\overset{+}{\Delta}) + 4\mu\overset{+}{\Delta}/3 = \hat{\sigma}(\Delta) + 4\mu\Delta/3 \tag{32}$$

which, in terms of the uni-axial deformation response function $\Sigma(\Delta)$, reads

$$\Sigma(\overset{+}{\Delta}) = \Sigma(\Delta). \tag{33}$$

Next we will show that if the (necessary) condition (33) holds, then this in fact guarantees the existence of a solution to the shock problem. In order to show this, suppose that $\overset{+}{\mathbf{H}}$ is a given tensor. Define $\overset{+}{\Delta}$ by $(21)_1$, $(20)_1$. If there exists a number $\Delta$ ($\neq \overset{+}{\Delta}$) such that (33) holds, then (for *each* arbitrary unit vector **n**) we can define $\alpha$ by (30), **a** by (29) and **H** by (24). It may be readily verified that these tensors $\overset{+}{\mathbf{H}}$ and **H** automatically satisfy the requirements (22), (23) of displacement and traction continuity across the plane with unit normal **n**. Thus we have the following result:

*Proposition.* Given a tensor $\overset{+}{\mathbf{H}}$, there exists an associated piecewise homogeneous equilibrium shock if and only if there is a number $\Delta$ ($\neq \overset{+}{\Delta} \equiv \overset{+}{H}_{kk}$) such that (33) holds.

When the constitutive law is such that the stress response function $\Sigma(\varepsilon)$ is monotonically increasing (in which case the material is elliptic) we see from the preceding proposition that the material cannot sustain a piecewise homogeneous deformation. On the other hand, if the material is such that $\Sigma'(\Delta) \leqslant 0$ on some interval, then since $\Sigma'(0) > 0$, it follows that piecewise homogeneous deformations will exist for suitably chosen values of $\overset{+}{\mathbf{H}}$.

In addition to providing information concerning the existence of a piecewise homogeneous deformation associated with the given displacement gradient $\overset{+}{\mathbf{H}}$, the preceding result also permits us to characterize the set of all such deformations which can be associated with that $\overset{+}{\mathbf{H}}$: it states that for every number $\bar{\Delta}$ for which (33) holds, and for *all* choices of the unit normal vector $\mathbf{n}$, one can construct an acceptable $\bar{\mathbf{H}}$. Let $\Xi$ denote the following set in the $(\overset{+}{\Delta}, \bar{\Delta})$-plane:

$$\Xi = \{(\overset{+}{\Delta}, \bar{\Delta}) \mid \Sigma(\overset{+}{\Delta}) = \Sigma(\bar{\Delta}), \overset{+}{\Delta} \neq \bar{\Delta}\}. \tag{34}$$

According to the preceding proposition, given a displacement gradient tensor $\overset{+}{\mathbf{H}}$, the associated shock problem has a solution if and only if there is a number $\bar{\Delta}$ such that $(\overset{+}{\Delta}, \bar{\Delta}) \in \Xi$ where $\overset{+}{\Delta} = \overset{+}{H}_{kk}$; moreover, all tensors $\mathbf{H}$ that can be connected to $\overset{+}{\mathbf{H}}$ by a shock are generated by all numbers $\bar{\Delta}$ for which $(\overset{+}{\Delta}, \bar{\Delta}) \in \Xi$. The set $\Xi$ characterizes the collection of all possible shocks. A sketch of the curve $\Xi$ in the $(\overset{+}{\Delta}, \bar{\Delta})$-plane, corresponding to a particular class of materials, will be given in Section 5.

Finally, we note that according to (24), (29), (30) the displacement gradient tensors $\overset{+}{\mathbf{H}}$ and $\mathbf{H}$ are related by

$$H_{ij} = \overset{+}{H}_{ij} + (\Delta - \overset{+}{\Delta}) n_i n_j, \tag{35}$$

this implies that the deformation on $R$ is equivalent to the deformation on $\overset{+}{R}$ together with a uni-axial stretch in the direction normal to the shock surface. This is presumably the reason why the stress response function in uni-axial deformation $\Sigma(\varepsilon)$ plays such a central role in the preceding (and subsequent) results.

## 4. DRIVING TRACTION

We now consider a quasi-static motion of the body and let $u(\cdot, t)$, $t_0 \leqslant t \leqslant t_1$, be a one-parameter family of solutions of the displacement equations of equilibrium (13) of the type described in Section 2. Let $S_t \subset R$ be the family of shocks associated with this motion, and assume that the particle velocity $\mathbf{v}(\mathbf{x}, t) = \partial \mathbf{u}(\mathbf{x}, t)/\partial t$ exists and is continuous in $(\mathbf{x}, t)$ for $\mathbf{x} \in R - S_t$, $t_0 \leqslant t \leqslant t_1$, and that $\mathbf{v}$ is piecewise continuous on $R \times [t_0, t_1]$.

Let $\mathrm{d}(t)$ denote the difference between the rate of external work (on any fixed regular region $\Pi \subset R$) and the rate at which elastic energy is being stored (in $\Pi$):

$$d(t) = \int_{\partial\Pi} \sigma_{ij} n_j v_i \, \mathrm{d}A - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Pi} W(\varepsilon) \, \mathrm{d}V, \quad t_0 \leqslant t \leqslant t_1; \tag{36}$$

$\mathrm{d}(t)$ is the *rate of dissipation* of mechanical energy in the region $\Pi$. By adapting to the present small-strain theory the analysis given by Knowles (1983), one can show that $\mathrm{d}(t)$ may be written as

$$\mathrm{d}(t) = \int_{S_t \cap \Pi} f \mathbf{n} \cdot \mathbf{V} \, \mathrm{d}A \tag{37}$$

where $f(\mathbf{x}, t)$ is defined by

$$f = [[\mathbf{P}]]^{+} \mathbf{n} \cdot \mathbf{n}, \quad \text{for} \quad \mathbf{x} \in S_t, \quad t_0 \leqslant t \leqslant t_1, \tag{38}$$

$\mathbf{P}(\mathbf{x}, t)$ is the energy–momentum tensor

$$P_{ij} = W(\varepsilon)\delta_{ij} - \sigma_{kj} H_{ki} \quad \text{for} \quad \mathbf{x} \in R - S_t, \quad t_0 \leqslant t \leqslant t_1 \tag{39}$$

and $\mathbf{V}(\mathbf{x}, t)$ is the velocity of a point on the moving surface $S_t$. If the motion happens to be smooth, $\mathbf{P}(\cdot, t)$ will be continuous across $S_t$ and so (37), (38) gives $\mathrm{d}(t) = 0$ for $t_0 \leqslant t \leqslant t_1$. In general however the dissipation rate $\mathrm{d}(t) \neq 0$ whenever $\Pi$ intersects $S_t$.

Combining (36) with (37) yields

$$\int_{\partial\Pi} \sigma\mathbf{n} \cdot \mathbf{v} \, \mathrm{d}A + \int_{S_t \cap \Pi} (-f\mathbf{n}) \cdot \mathbf{V} \, \mathrm{d}A = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Pi} W(\varepsilon) \, \mathrm{d}V, \quad t_0 \leqslant t \leqslant t_1 \tag{40}$$

which may be viewed as a work–energy identity. It states that the sum of the rates at which work is being done on $\Pi$ by the external forces and the phase boundary $S_t$ balances the rate at which energy is being stored in $\Pi$. Accordingly, $-f\mathbf{n}$ may be thought of as the traction applied by the surface $S_t$ on the body, or equivalently, $+f\mathbf{n}$ can be viewed as a "*driving traction*" exerted on the phase boundary $S_t$ by the surrounding material; the scalar $f$ determines the magnitude of this traction. The expression (38) (with (39)) is a special case of a formula given by Eshelby (1970); see also Eshelby (1956), Rice (1975).

If we postulate that at each instant, the rate of storage of energy in $\Pi$ cannot exceed the rate of external work on $\Pi$, then we must require the dissipation rate $\mathrm{d}(t)$ to be non-negative for all sub-regions $\Pi$ and all instants $t$. Thus, by (37),

$$f V_n \geqslant 0 \quad \text{for} \quad \mathbf{x} \in S_t, \quad t_0 \leqslant t \leqslant t_1 \tag{41}$$

where $V_n$ is the normal velocity of a point on the surface $S_t$:

$$V_n = \mathbf{V} \cdot \mathbf{n} \quad \text{for} \quad \mathbf{x} \in S_t, \quad t_0 \leqslant t \leqslant t_1. \tag{42}$$

Alternatively, the *dissipation inequality* (41) can be shown to be a consequence of the second law of thermodynamics under isothermal conditions; see Knowles (1979). In general, given an equilibrium state, the inequality (41) restricts the direction in which the surface $S_t$ may move in a quasi-static motion commencing from this state.

A particularly simple expression for the driving traction can be derived in the case of materials characterized by the special elastic potential (9). First, from (24), (29) one has

$$[[H_{ki}]]^{+} = -\alpha n_i n_k. \tag{43}$$

Next, in view of (10), (12), (43) and the continuity of traction,

$$[[\sigma_{kj} H_{ki} n_i n_j]]^{+} = -\alpha\{2\mu \overset{+}{\varepsilon}_{ij} n_i n_j - 2\mu\overset{+}{\Delta} + \Sigma(\overset{+}{\Delta})\}. \tag{44}$$

However from (3), (1) and (31) one obtains

$$[[k^2]]^{+} = -4\alpha \overset{+}{\varepsilon}_{ij} n_i n_j - 2\alpha^2 + 2\alpha(\overset{+}{\Delta} + \bar{\Delta})/3 \tag{45}$$

which can be used to eliminate the term $\overset{+}{\varepsilon}_{ij} n_i n_j$ from (44) to give

$$[[\sigma_{kj}H_{ki}n_in_j]]^+_- = \mu(\overset{+}{k}^2-\bar{k}^2)/2+2\mu(\bar{\Delta}^2-\overset{+}{\Delta}^2)/3-\alpha\Sigma(\overset{+}{\Delta}). \tag{46}$$

Finally, since (9) and (12) provide

$$W(\varepsilon) = \mu k^2/2-2\mu\Delta^2/3+\int_0^{\Delta}\Sigma(\xi)\,\mathrm{d}\xi, \tag{47}$$

eqns (38), (39), (46), (47) and (30) yield the desired expression

$$f = \int_{\bar{\Delta}}^{\overset{+}{\Delta}}\Sigma(\Delta)\,\mathrm{d}\Delta-\Sigma(\overset{+}{\Delta})(\overset{+}{\Delta}-\bar{\Delta}) \quad \text{for} \quad \mathbf{x}\in S_t, \quad t_0\leqslant t\leqslant t_1 \tag{48}$$

for the driving traction. In the finite theory, formulae of this general form have been derived in the special case of "normal shocks" in plane and anti-plane finite deformations of isotropic, incompressible elastic solids (Abeyaratne and Knowles, 1989; Yatomi and Nishimura, 1983).

It is useful to write (48) as

$$f = F(\overset{+}{\Delta},\bar{\Delta}) \quad \text{for} \quad \mathbf{x}\in S_t, \quad t_0\leqslant t\leqslant t_1 \tag{49}$$

where $F$ is the function defined on the set $\Xi$ by

$$F(\overset{+}{\Delta},\bar{\Delta}) = \int_{\bar{\Delta}}^{\overset{+}{\Delta}}\Sigma(\Delta)\,\mathrm{d}\Delta-\Sigma(\overset{+}{\Delta})(\overset{+}{\Delta}-\bar{\Delta}) \quad \text{for} \quad (\overset{+}{\Delta},\bar{\Delta})\in\Xi. \tag{50}$$

By (49), the driving traction $f$ at a point on the phase boundary $S_t$ depends only on the local dilatations $\overset{+}{\Delta}$, $\bar{\Delta}$ on the two sides of $S_t$; $f$ does not depend on the amounts of shear $\overset{+}{k}$, $\bar{k}$, nor on the orientation of $S_t$. Moreover, in view of (50) and (33), the value of $f$ may be interpreted geometrically as the difference between the area under the uni-axial deformation stress–strain curve between $\bar{\Delta}$ and $\overset{+}{\Delta}$, and the area of the rectangle on the same base with height $\Sigma(\overset{+}{\Delta})$.

## 5. KINETIC LAW. AN EXAMPLE

In Part II (Abeyaratne and Jiang, 1989) we will present an example which shows that boundary-value problems formulated in the conventional manner, for materials characterized by (9), may suffer from a tremendous lack of uniqueness. This is known to be the case in the finite theory as well (e.g. Abeyaratne, 1980; Abeyaratne and Knowles, 1989). This non-uniqueness suggests that the theory, as formulated, is deficient, and that it ought to be supplemented with additional constitutive information. One way in which to implement this is to postulate a constitutive relation, or "kinetic law", which applies to particles on $S_t$, and relates the driving traction $f$ to the normal velocity of propagation $V_n$ of the phase boundary.

In order to formalize this, let $f_M$ and $f_m$ be the supremum and infemum of the function $F(\overset{+}{\Delta},\bar{\Delta})$ on the set $\Xi$. Then, one might suppose that there is a constitutive function $V(\cdot)$ defined on $[f_m, f_M]$ such that

$$V_n = V(f) \quad \text{on} \quad S_t, \quad t_0\leqslant t\leqslant t_1. \tag{51}$$

In order to conform to the dissipativity inequality (42), $V$ must be such that

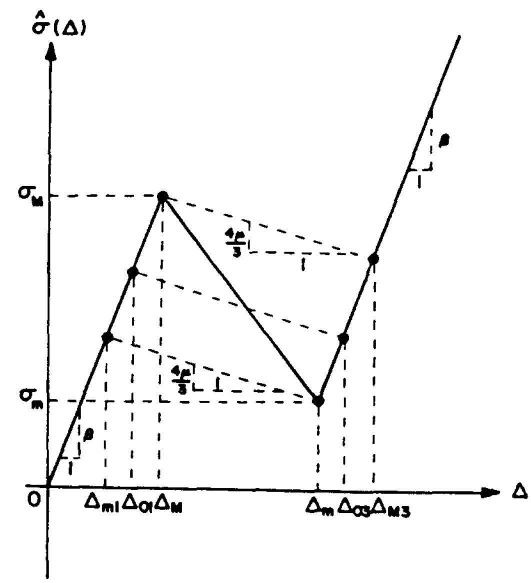


Fig. 1. Stress response curve in pure dilatation.

$$fV(f) \geqslant 0 \quad \text{for} \quad f \in [f_m, f_M]. \tag{52}$$

The form (51) is, of course, merely an example of a class of kinetic laws that might be imposed; it could be generalized to include dependence on other local variables as well so that, for example, a kinetic law might read $V_n = V(\hat{\Delta}, \bar{\Delta})$, where the constitutive function $V$ is defined on $\Xi$.

In order to illustrate this (and some of the preceding results) we now choose the dilatational stress response function in the constitutive law (10) to be as follows

$$\hat{\sigma}(\Delta) = \begin{cases} \beta\Delta & \text{for } 0 \leqslant \Delta \leqslant \Delta_M \\ \beta\Delta + \sigma_T(\Delta - \Delta_M)/(\Delta_m - \Delta_M) & \text{for } \Delta_M \leqslant \Delta \leqslant \Delta_m \\ \beta\Delta + \sigma_T & \text{for } \Delta \geqslant \Delta_m\,; \end{cases} \tag{53}$$

$\beta$, $\Delta_m$, $\Delta_M$ and $\sigma_T$ are material constant such that

$$\left.\begin{array}{l} \beta > 0, \quad \Delta_m > \Delta_M > 0, \quad \sigma_T < 0 \\ \beta\Delta_m + \sigma_T > 0 \\ (\Delta_m - \Delta_M)(\beta + 4\mu/3) < -\sigma_T. \end{array}\right\} \tag{54}$$

The second condition in (54) implies that $\hat{\sigma}(\Delta_m) > 0$, while $(54)_3$ ensures that the system of equations (13) is non-elliptic when $\Delta_m < \Delta(x) < \Delta_M$. In this example we will confine attention to the range $\Delta > 0$ and consequently we have left $\hat{\sigma}(\Delta)$ undefined for negative values of its argument. The specific constitutive law (53), (54) is the one considered by Budiansky, *et al.* (1983) in the case of super-critical transformations. As shown in Fig. 1, as the dilatation increases, the mean stress first rises linearly to a maximum value $\sigma_M = \beta\Delta_M$, it then declines linearly to the value $\sigma_m = \beta\Delta_m + \sigma_T$, and finally rises again with the initial slope $\beta$.

The response function of this material in uni-axial deformations is given by (53), (12) as

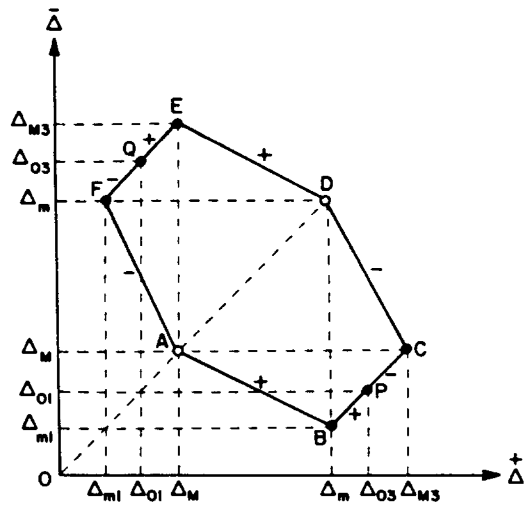


Fig. 2. The set $\Xi$ characterizing all possible shock states.

$$\Sigma(\varepsilon) = \begin{cases} \alpha\varepsilon & \text{for } 0 \leqslant \varepsilon \leqslant \Delta_M \\ \alpha\varepsilon + \sigma_T(\varepsilon - \Delta_M)/(\Delta_m - \Delta_M) & \text{for } \Delta_M \leqslant \varepsilon \leqslant \Delta_m \\ \alpha\varepsilon + \sigma_T & \text{for } \varepsilon \geqslant \Delta_m \end{cases} \tag{55}$$

where we have set

$$\alpha = \beta + 4\mu/3. \tag{56}$$

Finally, we introduce the following additional notation which pertains to certain special points on the stress–strain curve shown in Fig. 1:

$$\left.\begin{aligned} \Delta_{m1} &= \Delta_m + \sigma_T/\alpha, & \Delta_{M3} &= \Delta_M - \sigma_T/\alpha \\ \Delta_{o1} &= (\Delta_m + \Delta_M)/2 + \sigma_T/2\alpha, & \Delta_{o3} &= (\Delta_m + \Delta_M)/2 - \sigma_T/2\alpha. \end{aligned}\right\} \tag{57}$$

Note that the straight lines which join $(\Delta_M, \sigma_M)$ to $(\Delta_{M3}, \hat{\sigma}(\Delta_{M3}))$, $(\Delta_{o1}, \hat{\sigma}(\Delta_{o1}))$ to $(\Delta_{o3}, \hat{\sigma}(\Delta_{o3}))$, and $(\Delta_{m1}, \hat{\sigma}(\Delta_{m1}))$ to $(\Delta_m, \sigma_m)$, each have the same slope, $-4\mu/3$; see (32) for the significance of this. Moreover, $\Delta_{o1}$ and $\Delta_{o3}$ are seen to obey the conditions

$$\left.\begin{aligned} &\Sigma(\Delta_{o1}) = \Sigma(\Delta_{o3}) = (\Sigma_m + \Sigma_M)/2 \\ \text{where} \quad &\Sigma_m = \Sigma(\Delta_m), \quad \Sigma_M = \Sigma(\Delta_M). \end{aligned}\right\} \tag{58}$$

The set $\Xi$ for this material, which characterizes the complete set of possible shocks in the $(\overset{+}{\Delta}, \bar{\Delta})$-plane, may be readily found from (34), (55). It consists of the points *on* the polygon ABCDEFA shown in Fig. 2, except for the vertices A and D which lie on the line $\overset{+}{\Delta} = \bar{\Delta}$.

We turn next to equation (50) which defines the driving traction function $F$ on this set $\Xi$. Explicit formulae for $F$ may be readily derived from (50), (55). For example, when $(\overset{+}{\Delta}, \bar{\Delta}) \in EF$, one finds

$$F(\overset{+}{\Delta}, \bar{\Delta}) = (-\sigma_T/\alpha)\{\alpha\overset{+}{\Delta} - (\Sigma_m + \Sigma_M)/2\}. \tag{59}$$

We do not display the remaining formulae here. It is particularly useful to know the sign of the driving traction, since then the direction of propagation of the phase boundary is known through the dissipativity inequality (41). The sign of $F$ may be read off from (59) (and the analogous formulae appropriate to the other points on $\Xi$); one finds that

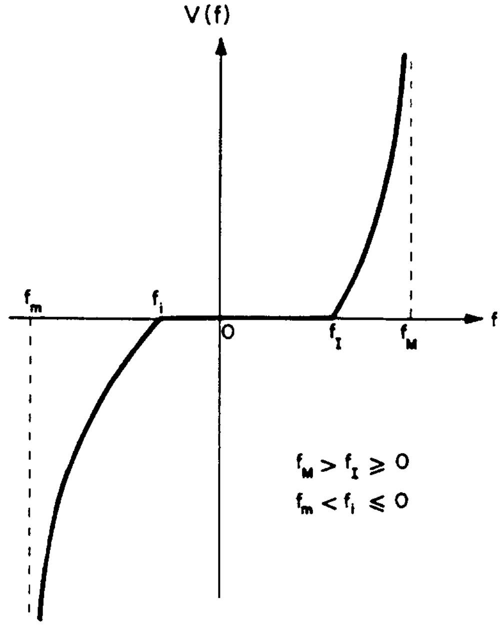


Fig. 3. Kinetic response function.

$$F(\overset{+}{\Delta},\overset{-}{\Delta}) = \begin{cases} > 0 & \text{for } (\overset{+}{\Delta},\overset{-}{\Delta}) \in (\mathrm{AB}]+[\mathrm{BP})+(\mathrm{DE}]+[\mathrm{EQ}), \\ < 0 & \text{for } (\overset{+}{\Delta},\overset{-}{\Delta}) \in (\mathrm{PC}]+[\mathrm{CD})+(\mathrm{QF}]+[\mathrm{FA}), \\ = 0 & \text{for } (\overset{+}{\Delta},\overset{-}{\Delta}) = \mathrm{P} \text{ or } \mathrm{Q}. \end{cases} \tag{60}$$

(The symbol (AB] in (60) denotes the set of all points on the line AB excluding the end point A but including the point B.) The points P and Q which are associated with zero driving traction are sometimes referred to as "Maxwell states". They are given by

$$(\overset{+}{\Delta},\overset{-}{\Delta}) = (\Delta_{o3},\Delta_{o1}) \quad \text{and} \quad (\Delta_{o1},\Delta_{o3}) \tag{61}$$

where $\Delta_{o1}$ and $\Delta_{o3}$ were defined in (57); see also (58). Also, one finds that the driving traction achieves its largest value $f_M$ at B (and also at E) and its smallest value $f_m$ at C (and also at F). These values are

$$f_M = -\sigma_T(\Sigma_M-\Sigma_m)/2\alpha \quad (>0) \tag{62}$$

$$f_m = \sigma_T(\Sigma_M-\Sigma_m)/2\alpha \qquad (<0). \tag{63}$$

Finally, Fig. 3 shows an example of a kinetic function $V$ that might be used in the kinetic law (51). It is consistent with the admissibility requirement (52). In the example which will be discussed in Part II, we will see how in a specific problem, the kinetic relation, together with an initiation criterion, can be used to resolve the non-uniqueness referred to earlier. In that example we will find that the kinetic relation of Fig. 3 generally leads to rate-dependent "viscoplasticity-like" response. Two special cases which lead to reversible, dissipation-free response and to rate-independent plasticity-like response will also be discussed there.

### 5.1. *Some remarks on the kinetic law*

In this sub-section we make some further observations on the kinetic relation. Consider a deformed configuration of the body involving a shock $S$. Let $\mathbf{x}$ be a point on $S$ and let $\overset{+}{\varepsilon}$

and $\overset{+}{\boldsymbol{\sigma}}$ be the limiting values of the strain and stress tensors at $\mathbf{x}$ as $\mathbf{x}$ is approached from the positive side of $S$; $\overset{-}{\boldsymbol{\varepsilon}}$ and $\overset{-}{\boldsymbol{\sigma}}$ are the corresponding quantities on the negative side. The unit normal to $S$ at $\mathbf{x}$, which points into the positive side, is $\mathbf{n}$.

Let $A$ denote the set of all symmetric tensors $\boldsymbol{\varepsilon}$ which are "kinematically compatible" with $\overset{+}{\boldsymbol{\varepsilon}}$, $\overset{-}{\boldsymbol{\varepsilon}}$ and $\mathbf{n}$, i.e. tensors $\boldsymbol{\varepsilon}$ such that the pair of strain tensors $\boldsymbol{\varepsilon}$, $\overset{+}{\boldsymbol{\varepsilon}}$ (or $\boldsymbol{\varepsilon}$, $\overset{-}{\boldsymbol{\varepsilon}}$) can be associated with a piecewise homogeneous deformation which has continuous displacements across the surface with normal $\mathbf{n}$:

$$A = \{\boldsymbol{\varepsilon} \,|\, \varepsilon_{ij} = \overset{+}{\varepsilon}_{ij} + (1/2)(a_i n_j + a_j n_i) \quad \text{for an arbitrary vector } \mathbf{a}\}. \tag{64}$$

(Since $\overset{+}{\boldsymbol{\varepsilon}}$ and $\overset{-}{\boldsymbol{\varepsilon}}$ are related by (25), the set $A$ defined by (64) with $\overset{+}{\boldsymbol{\varepsilon}}$ is identical to the set corresponding to $\overset{-}{\boldsymbol{\varepsilon}}$.) Clearly, both $\overset{+}{\boldsymbol{\varepsilon}}$ and $\overset{-}{\boldsymbol{\varepsilon}} \in A$. Since $\overset{+}{\boldsymbol{\sigma}}$ and $\overset{-}{\boldsymbol{\sigma}}$ obey (5), it is not difficult to show that

$$\overset{+}{\sigma}_{ij}[\varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(2)}] = \overset{-}{\sigma}_{ij}[\varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(2)}] \quad \text{for all } \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)} \in A. \tag{65}$$

Next, we defined a function $G$ by

$$G(\boldsymbol{\varepsilon}) = W(\boldsymbol{\varepsilon}) - (1/2)(\overset{+}{\sigma}_{ij} + \overset{-}{\sigma}_{ij})\varepsilon_{ij} \tag{66}$$

for all tensors $\boldsymbol{\varepsilon}$ in $A$. In order to study the extrema of $G$, it is easier to consider the function $\hat{G}$ which is defined on the set of all *vectors* and is such that

$$\hat{G}(\mathbf{a}) = G(\boldsymbol{\varepsilon}) \tag{67}$$

with $\boldsymbol{\varepsilon}$ and $\mathbf{a}$ related by (64). Differentiating $\hat{G}$ with respect to $a_i$ yields

$$\frac{\partial \hat{G}}{\partial a_i} = \left(\frac{\partial W(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}} - (1/2)(\overset{+}{\sigma}_{ij} + \overset{-}{\sigma}_{ij})\right) n_j \tag{68}$$

which when differentiated once more gives

$$\frac{\partial^2 \hat{G}}{\partial a_i\, \partial a_j} = \frac{\partial^2 W(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ip}\, \partial \varepsilon_{jq}} n_p n_q. \tag{69}$$

It now follows from (68), (5) and the constitutive relationships $\overset{+}{\sigma}_{ij} = \partial W(\overset{+}{\boldsymbol{\varepsilon}})/\varepsilon_{ij}$, $\overset{-}{\sigma}_{ij} = \partial W(\overset{-}{\boldsymbol{\varepsilon}})/\varepsilon_{ij}$, that $\boldsymbol{\varepsilon} = \overset{+}{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\varepsilon} = \overset{-}{\boldsymbol{\varepsilon}}$ are both extrema of $G$, and from (69) and (15) that they (locally) minimize $G$ if strong ellipticity prevails at the appropriate strain $\overset{+}{\boldsymbol{\varepsilon}}$ or $\overset{-}{\boldsymbol{\varepsilon}}$.

Finally, on using (65), (66), (5), (38) and (39) we see that the difference between the values of $G$ on the two sides of $S$ is precisely the driving traction on $S$ at $\mathbf{x}$:

$$f = G(\overset{+}{\boldsymbol{\varepsilon}}) - G(\overset{-}{\boldsymbol{\varepsilon}}). \tag{70}$$

In order to illustrate this, consider the straight line in strain space,

$$\varepsilon_{ij} = \frac{\Delta - \overset{-}{\Delta}}{\overset{+}{\Delta} - \overset{-}{\Delta}} \overset{+}{\varepsilon}_{ij} + \frac{\overset{+}{\Delta} - \Delta}{\overset{+}{\Delta} - \overset{-}{\Delta}} \overset{-}{\varepsilon}_{ij} \quad \text{for } -\infty < \Delta < \infty \tag{71}$$

which connects the strains $\overset{+}{\boldsymbol{\varepsilon}}$ and $\overset{-}{\boldsymbol{\varepsilon}}$. All tensors $\boldsymbol{\varepsilon}$ on this line belong to the set $A$. Let $\overset{0}{G}(\Delta)$ be the restriction of $G(\boldsymbol{\varepsilon})$ to this line. Then, Fig. 4 is a schematic diagram showing a graph

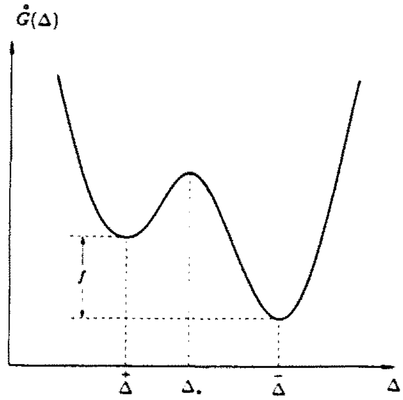


Fig. 4. Schematic graph of the function $\overset{0}{G}(\Delta)$.

of $\overset{0}{G}(\Delta)$ versus $\Delta$, in a case where strong ellipticity prevails at $\overset{+}{\varepsilon}$ and $\bar{\varepsilon}$ (i.e. at $\Delta = \overset{+}{\Delta}$ and $\bar{\Delta}$) so that they both correspond to minima of $G$. As the shock surface $S$ moves through the body, its normal velocity $V_n$ at x is a measure of the (volume) rate of change at which material on the positive side is transferred to the negative side. *The kinetic relation* (51) *is therefore a relationship between this rate and the jump in G.*

The discussion so far in this sub-section has not been specialized to the material (9). In the case when the material is characterized by the strain energy function (9), one finds by substituting (9) and (10) into (66), and simplifying using (71), (12), (1), (3), that

$$\overset{0}{G}(\Delta) = \int_0^{\Delta} \Sigma(\xi)\, d\xi - (1/2)[\Sigma(\overset{+}{\Delta}) + \Sigma(\Delta)]\Delta + \text{constant}, \quad -\infty < \Delta < \infty. \tag{72}$$

When the response function in dilatation is as depicted in Fig. 1, it is easy to show that the graph of $\overset{0}{G}$ as given by (72), (53), has the general form shown in Fig. 4; when $\Sigma_m < \Sigma(\overset{+}{\Delta}) < \Sigma_M$, $\overset{0}{G}$ has precisely two local minima and one local maximum as in Fig. 4. Moreover, *in this case*, $G(\overset{+}{\Delta}) - G(\bar{\Delta})$ (and therefore the driving traction $f$) is equal to the difference in the *Gibbs free energy* $[[W(\varepsilon) - \sigma_{ij}\varepsilon_{ij}]]_-^+$.

Diagrams of the form of Fig. 4 are commonly encountered in metallurgical discussions on the kinetics of phase transformations; see for example Porter and Easterling (1981). The classical example of a kinetic relation in this context is the Arhhenius law which, in the notation of Fig. 4, is based on the assumption that the rate of transfer of material from the positive side of $S$ to the negative side is governed by the quantity $\overset{0}{G}(\Delta_*) - \overset{0}{G}(\overset{+}{\Delta})$(which is the "height of the barrier" in Fig. 4) and that the rate of transfer of material from the negative side of $S$ to the positive side is governed by the quantity $\overset{0}{G}(\Delta_*) - \overset{0}{G}(\bar{\Delta})$. This leads to an explicit kinetic law of the form $V_n = V(f)$. We refer to Fine (1964) (which is based on Turnbull, 1956) for the details of such an argument.

## 6. CONCLUDING REMARKS

### 6.1. *Driving force on a crack-tip*

In this section we briefly comment on the driving force on a crack-tip when the crack is contained in a body composed of the material (10). For simplicity, suppose that the body is a slab containing a traction-free through-crack (Fig. 5) and that the loading is such that the deformation is planar. Suppose further that the body is composed of the material (10) with the constitutive function $\Sigma(\varepsilon)$ defined by (12) being non-monotone. By the analysis in Section 3, deformations of this body can involve shocks. Suppose for definiteness that there is a single (cylindrical) phase boundary $S$ as shown in Fig. 4; $C$ is the curve along which $S$

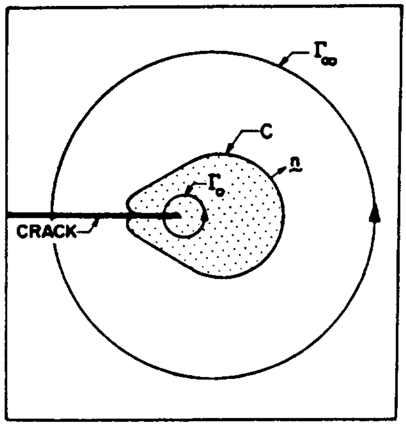


Fig. 5. Geometry of cracked slab with phase boundary $C$ and integration paths $\Gamma_0$, $\Gamma_\infty$.

intersects the $(x_1, x_2)$-plane. The deformation is smooth at all points of the body inside $C$ (excluding points on the crack itself) as well as at all points outside $C$.

Let $\Gamma_0$ and $\Gamma_\infty$ be two closed curves as shown in Fig. 5 with $\Gamma_0$ being entirely within $C$ and $\Gamma_\infty$ entirely outside. The values of the $J$-integral associated with these two curves are, respectively,

$$J_{\text{tip}} \equiv \oint_{\Gamma_0} P_{1\beta} n_\beta \, \mathrm{d}s, \quad J_\infty \equiv \oint_{\Gamma_\infty} P_{1\beta} n_\beta \, \mathrm{d}s \tag{73}$$

where d$s$ denotes arc length, $\mathbf{n}$ is the unit outward normal vector on the appropriate curve, and $P_{\alpha\beta}$ are the components of the energy–momentum tensor:

$$P_{\alpha\beta} \equiv W\delta_{\alpha\beta} - \sigma_{\gamma\beta} H_{\gamma\alpha}. \tag{74}$$

(Greek subscripts take the values 1 and 2 only.)

The $J$-integral is path-independent *provided the paths of integration do not insect the shock curve $C$*; this, together with the traction-free nature of the crack surface yields the alternate expression

$$J_{\text{tip}} = \oint_C \bar{P}_{1\beta} n_\beta \, \mathrm{d}s \tag{75}$$

where $\bar{P}_{\alpha\beta}$ are the limiting values of $P_{\alpha\beta}$ as a point on $C$ is approached from within. Similarly

$$J_\infty = \oint_C \overset{+}{P}_{1\beta} n_\beta \, \mathrm{d}s. \tag{76}$$

Combining (75) and (76) gives

$$J_{\text{tip}} = J_\infty - \oint_C [[P_{1\beta}]]_-^+ n_\beta \, \mathrm{d}s. \tag{77}$$

Next, in view of (74), (1), displacement continuity (2), and traction continuity (5), one sees that

$$[[P_{\alpha\beta}]]_-^+ n_\beta l_\alpha = 0 \quad \text{on } C \tag{78}$$

where $\ell$ is a unit tangent vector on $C$. Thus the vector $[[P_{\alpha\beta}]]_-^+ n_\beta$ is normal to the curve $C$, and by (38)

$$[[P_{\alpha\beta}]]_-^+ n_\beta = f n_\beta \quad \text{on } C; \tag{79}$$

$f$ is the driving traction on the shock. Finally, combining (77) with (79) provides the desired expression

$$J_{\text{tip}} = J_\infty - \oint_C f n_1 \, ds. \tag{80}$$

Equation (78) states that the driving force $J_{\text{tip}}$ on the crack-tip equals the difference between $J_\infty$ (the "applied value of $J$") and the resultant driving force on the shock. Thus in general, $J_{\text{tip}} \neq J_\infty$. (This was also noted by Silling, 1987.) In certain exceptional cases, for example if the deformation is such that $f$ = constant on $C$, the integral in (80) will vanish and then $J_{\text{tip}} = J_\infty$. The value of the shock driving traction $f$ depends on (and is determined by) the particular kinetic relation governing the evolution of the shock. If the resultant driving force on the shock is in the positive $x_1$-direction then, by (80), $J_{\text{tip}} < J_\infty$.

### 6.2. *Anti-plane deformations and the constitutive law*

Consider a right-cylindrical body whose middle cross-section $D$ lies in the $(x_1, x_2)$-plane, and suppose that a purely axial displacement field $f(x_1, x_2)$ is prescribed on the lateral surface of the body:

$$u_1 = u_2 = 0, \quad u_3 = f(x_1, x_2) \quad \text{on } \partial D. \tag{81}$$

Assume, for the moment, that the cylinder is composed of a homogeneous, isotropic elastic material characterized by a strain energy function $W(k, \Delta)$ which is *not* necessarily of the particular form (9).

In order to examine whether this body can respond to the loading (81) in an anti-plane manner, we assume that it does and take

$$u_1 = u_2 = 0, \quad u_3 = u(x_1, x_2) \quad \text{on } D \tag{82}$$

with $u = f$ on $\partial D$. From (1) and (3), we find that the shear and dilatational invariants associated with the displacement field (82) are

$$k = |\nabla u|, \quad \Delta = 0 \quad \text{on } D. \tag{83}$$

By (7), the corresponding stress components are

$$\left.\begin{aligned} \sigma_{11} = \sigma_{22} = \sigma_{33} = \partial W(k,0)/\partial\Delta, \quad & \sigma_{12} = 0 \\ \sigma_{13} = (u_{,1}/k)\, \partial W(k,0)/\partial k, \quad & \sigma_{23} = (u_{,2}/k)\, \partial W(k,0)/\partial k \end{aligned}\right\} \quad \text{on } D \tag{84}$$

with $k$ given by (83). Substituting (84) into the equilibrium equations leads to the following *three* equations involving the *single* displacement component $u$:

$$\left.\begin{aligned} &\frac{\partial}{\partial x_1}[\partial W(k,0)/\partial\Delta] = 0, \quad \frac{\partial}{\partial x_2}[\partial W(k,0)/\partial\Delta] = 0 \\ &\frac{\partial}{\partial x_1}[(u_{,1}/k)\,\partial W(k,0)/\partial k] + \frac{\partial}{\partial x_2}[(u_{,2}/k)\,\partial W(k,0)/\partial k] = 0 \end{aligned}\right\} \text{ on } D \tag{85}$$

with $k = |\nabla u|$. In general, this is an over-determined system of equations and therefore has no solution.† Consequently, we conclude that *despite the purely axial nature of the prescribed boundary displacement* (81), *the displacement field within the body cannot (generally) be of the anti-plane form* (82). In particular, the in-plane displacement components $u_1$ and $u_2$ will not generally vanish, and neither will the dilatation.

If it so happens that the strain energy function $W$ has the particular (separable) form (8) or (9), then the first two equations in (85) are satisfied automatically and so the problem is then not over-determined. Therefore, in this special case, the body *can* deform in an anti-plane manner.

These observations may be of some relevance in the modeling of transforming ceramic composites. If a cylindrical body composed of such a material was loaded by a purely axial displacement (81), the second phase particles in the composite would undergo a martensitic transformation when the applied displacement became sufficiently large. Since this transformation involves some dilatation (which in fact is of primary interest in this setting) it follows that the displacement field within the body will not be of the anti-plane form (82). This in turn suggests that the strain energy function $W(k,\Delta)$ characterizing such materials might not have the separable form (8) or (9), but rather that the shear and dilatational dependency in $W$ would be coupled.

6.3. *Intersection of a shock with a traction-free surface*

Consider a shock surface $S$ which intersects the boundary $\partial D$ of the body. Let $\mathbf{x}$ be a point common to $S$ and $\partial D$, and suppose that the shock strength $\overset{+}{\Delta} - \bar{\Delta}$ does not vanish at $\mathbf{x}$. Let $\mathbf{m}$ denote the unit outward normal to $\partial D$ at $\mathbf{x}$ and let $\mathbf{n}$ be the (limiting) unit normal to $S$ at $\mathbf{x}$. Then, from (47) and (48) we have

$$\bar{\varepsilon}_{ij} = \overset{+}{\varepsilon}_{ij} + (\bar{\Delta} - \overset{+}{\Delta}) n_i n_j \tag{86}$$

while (26) and (28) give

$$\overset{+}{\sigma}_{ij} = 2\mu\,\overset{+}{\varepsilon}_{ij} + \{\Sigma(\overset{+}{\Delta}) - 2\mu\overset{+}{\Delta}\}\delta_{ij}, \quad \bar{\sigma}_{ij} = 2\mu\,\bar{\varepsilon}_{ij} + \{\Sigma(\bar{\Delta}) - 2\mu\bar{\Delta}\}\delta_{ij}. \tag{87}$$

Subtracting the second of (87) from the first, and using (86), leads to

$$\bar{\sigma}_{ij} = \overset{+}{\sigma}_{ij} + 2\mu(\bar{\Delta} - \overset{+}{\Delta})(n_i n_j - \delta_{ij}) \tag{88}$$

and so

$$\bar{\boldsymbol{\sigma}}\mathbf{m} = \overset{+}{\boldsymbol{\sigma}}\mathbf{m} + 2\mu(\bar{\Delta} - \overset{+}{\Delta})((\mathbf{n}\cdot\mathbf{m})\mathbf{n} - \mathbf{m}). \tag{89}$$

Suppose first that the boundary $\partial D$ is traction-free at $\mathbf{x}$ : $\overset{+}{\boldsymbol{\sigma}}\mathbf{m} = \bar{\boldsymbol{\sigma}}\mathbf{m} = \mathbf{0}$. Then (89) with $\bar{\Delta} \neq \overset{+}{\Delta}$ requires that

† Knowles (1977a, 1977b) has obtained a complete characterization of the class of materials which can sustain anti-plane shear deformations in the *finite* theory of elasticity.

$$(\mathbf{n}\cdot\mathbf{m})\mathbf{n} = \mathbf{m} \tag{90}$$

which implies that $\mathbf{m} = \pm\mathbf{n}$. Thus, the shock surface $S$ must be *tangential* to the boundary $\partial D$ at $\mathbf{x}$.

Suppose next that the boundary $\partial D$ is free of shear traction at $\mathbf{x}$. Then by (89),

$$(\mathbf{n}\cdot\mathbf{m})(\mathbf{n}\cdot\ell) = 0 \quad \text{for all vectors } \ell \text{ normal to } \mathbf{m}. \tag{91}$$

Thus in this case, either $\mathbf{m} = \pm\mathbf{n}$ or $\mathbf{m}$ is normal to $\mathbf{n}$ and so the shock surface $S$ is either *tangential or normal* to $D$ at $\mathbf{x}$. (Note that Fig. 5 is consistent with these properties: $C$ intersects the traction-free crack tangentially and the shear traction-free line ahead of the crack normally.)

*Acknowledgements*—The authors are grateful to John W. Hutchinson for having read and commented on this manuscript and to James K. Knowles and Stewart A. Silling for helpful conversations during the course of this work.

## REFERENCES


Abeyaratne, R. (1980). Discontinuous deformation gradients in plane finite elastostatics of incompressible materials. *J. Elasticity* **10**, 255–293.

Abeyaratne, R. (1981). Discontinuous deformation gradients in the finite twisting of an incompressible elastic tube. *J. Elasticity* **11**, 42–80.

Abeyaratne, R. (1983). An admissibility condition for equilibrium shocks in finite elasticity. *J. Elasticity* **13**, 175–184.

Abeyaratne, R. and Knowles, J. K. (1987). Non-elliptic elastic materials and the modeling of elastic-plastic behavior for finite deformation. *J. Mech. Phys. Solids* **35**, 343–365.

Abeyaratne, R. and Knowles, J. K. (1988). On the dissipative response due to discontinuous strains in bars of unstable elastic material. *Int. J. Solids Structures* **24**, 1021–1044.

Abeyaratne, R. and Knowles, J. K. (1989). Equilibrium shocks in plane deformations of incompressible elastic materials. *J. Elasticity* (to appear).

Abeyaratne, R. and Jiang, G.-H. (1989). Dilatationally nonlinear elastic materials—II. An example illustrating stress concentration reduction. *Int. J. Solids Structures* **25**, 1221–1233.

Budiansky, B., Hutchinson, J. W. and Lambropoulos, J. C. (1983). Continuum theory of dilatant transformation toughening in ceramics. *Int. J. Solids Structures* **19**, 337–355.

Chen, I. W. and Reyes Moral, R. E. (1986). Implications of transformation plasticity in $ZrO_2$ containing ceramics. I: Shear and dilatational effects. *J. Am. Ceramic Soc.* **69**, 181–189.

Ericksen, J. L. (1975). Equilibrium of bars. *J. Elasticity* **5**, 191–201.

Eshelby, J. D. (1970). Energy relations and the energy-momentum tensor in continuum mechanics. In *Inelastic Behavior of Solids* (Edited by M. F. Kanninen *et al.*). McGraw-Hill, New York.

Eshelby, J. D. (1956). The continuum theory of lattice defects. In *Solid State Physics* (Edited by F. Seitz and D. Turnbull), Vol. 3. Academic Press, New York.

Evans, A. G. and Cannon, R. (1986). Toughening of brittle solids by martensitic transformations. *Acta Mettal.* **34**, 761–800.

Fine, M. E. (1964). *Introduction to Phase Transformations in Condensed Systems.* Macmillan, New York.

Garvie, R. C., Hannink, R. H. and Pascoe, R. T. (1975). Ceramic steel? *Nature* **258**, 703–704.

Gurtin, M. E. (1983). Two-phase deformations of elastic solids. *Arch. Ratl Mech. Anal.* **84**, 1–29.

Hutchinson, J. W. and Neale, K. W. (1982). Finite strain $J_2$-deformation theory. In *Finite Elasticity* (Edited by D. E. Carlson and R. T. Shield). Martinus Nijhoff, The Hague.

James, R. D. (1984). The arrangement of coherent phases in a loaded body. In *Phase Transformations and Material Instabilities in Solids* (Edited by M. E. Gurtin), pp. 79–98. Academic Press, New York.

James, R. D. (1986). Displacive phase transformations in solids. *J. Mech. Phys. Solids* **34**, 359–394.

Knowles, J. K. (1977a). On finite anti-plane shear for incompressible elastic materials. *J. Aust. Math. Soc.*, Ser. B **19**, 400–415.

Knowles, J. K. (1977b). A note on anti-plane shear for compressible materials in finite elastostatics. *J. Aust. Math. Soc. Ser. B* **20**, 1–7.

Knowles, J. K. (1979). On the dissipation associated with equilibrium shocks in finite elasticity. *J. Elasticity* **9**, 131–158.

Knowles, J. K. and Sternberg, E. (1978). On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics. *J. Elasticity* **8**, 329–379.

Lambropoulos, J. (1986). Shear, shape and orientation effects in transformation toughening. *Int. J. Solids Structure* **22**, 1083–1106.

McMeeking, R. M. and Evans, A. G. (1982). Mechanics of transformation toughening in brittle materials. *J. Am. Ceramic Soc.* **65**, 242–246.

Porter, D. A. and Easterling, K. E. (1981). *Phase Transformations in Metals and Alloys.* Van Nostrand Reinhold, Scarborough, CA.

Rice, J. R. (1975). Continuum mechanics and thermodynamics of plasticity in relations to microscale deformation mechanisms. In *Constitutive Equations in Plasticity* (Edited by A. S. Argon). MIT Press, Cambridge.

Silling, S. A. (1987). The effect of supercritical dilatational transformation on crack-tip singularities. Brown University technical report.

Silling, S. A. (1988). Numerical studies of loss of ellipticity near singularities in an elastic material. *J. Elasticity* **19**, 213–239.

Turnbull, D. (1956). Phase changes. In *Solid State Physics* (Edited by F. Seitz and D. Turnbull), Vol. 3, pp. 225–306. Academic Press, New York.

Yatomi, C. and Nishimura, N. (1983). On the dissipation inequality in equilibrium shocks. *J. Elasticity* **13**, 311–316.